6. POLYNOMIALS

§6.1. Definition of a Polynomial

A **polynomial** is an expression of the form:

$$a(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

The symbol x is called an **indeterminate** and simply plays the role of a place marker. The role of the x is to provide positions in the expression which can be replaced (substituted) by a value. The numbers a_0, a_1,... are called the **coefficients** of the polynomial.

Forgive the corny pun. Polynomials have nothing to do with parrots. The word 'polynomial' comes from 'poly' in Greek, meaning 'many' as in polygons. and 'nomen' from Latin meaning 'name'. In mathematics 'nomial' refers to a term in an algebraic expression, as in the Binomial Theorem where $(a + b)$ in $(a + b)^n$ has two terms. So a polynomial is an algebraic expression with many terms. The term was first used in the 17th century.

You may be interested to know why parrots are often called "Polly". The name "Pol". or "Polly" for a parrot can be traced back to England since at least the early 1600s. In his 1606 comedy *Volpone*, playwright Ben Jonson described many of his characters as animals, reflecting their true nature. Two comic characters, Sir Politic Would-Be ("Sir Pol" for short) and his wife, are visitors from England who are trying to ingratiate themselves into Venetian society, and they try to speak the language of Venice by repeating words and phrases which they've heard but don't understand. Jonson describes them as "parrots".

Example 1: The expression $a(x) = x^3 - 4x^2 + 7x - 11$ is a polynomial in x. The coefficients of $a(x)$ are the numbers 1, –4, 7, –11.

The powers of x in a polynomial must be non-negative integers and there must be a finite number of terms. Other expressions have the appearance of being polynomials but, because some powers are negative or fractional, or because there are infinitely many terms, they are not considered to be polynomials.

Example 2: The expressions

$$x + \frac{1}{x}$$

$$1 - \sqrt{x} + x^2$$

$$\text{and } 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

are not polynomials even though they're built up from powers of x.

The coefficients can be rational numbers (forming the set $\mathbb{Q}$), real numbers (forming the set $\mathbb{R}$), or complex numbers (forming the set $\mathbb{C}$). These systems, called **fields**, have the property that x^{-1} exists for every non-zero x. (This somewhat loose description will do for our present purposes. An exact definition comprises 11 separate properties, or axioms.)

Sometimes we'll restrict our attention to polynomials with integer coefficients even though the integers don't form a field. Notice that a rational polynomial can always be converted to an integer polynomial by multiplying by a common denominator.

Example 3: The rational polynomial $\frac{22}{7}x^3 - \frac{5}{2}x + \frac{3}{4}$ is equal to $\dfrac{88x^3 - 70x + 21}{28}$.

It's also possible to consider polynomials whose coefficients are integers modulo p for some prime number. These systems are fields and the theory of polynomials over these finite fields works nicely even if some of the results look a bit strange.

For example over $\mathbb{Z}_2$, the field of integers modulo 2 (where there are just two elements, 0 and 1 with $1 + 1 = 0$) the factorisation of $x^2 + 1$ is $(x + 1)^2$, since the $2x$ term is equal to 0.

§6.2. Degree of a Polynomial

The **degree** of a polynomial is the largest power of x that occurs with a non-zero coefficient. That is, if:

$$a(x)=a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0,$$

the degree of $a(x)$ is n (provided that $a_n \neq 0$). We can write **deg** $a(x) = n$.

Examples 4: Polynomials of degree 2 are **quadratics**, of the form $ax^2 + bx + c$ (where $a \neq 0$).
Polynomials of degree 1 are the **linear polynomials** such as $2x + 3$ and $(½)x – ¼$.
Polynomials of degree 0 are the non-zero **constant polynomials**.

It might seem strange for numbers such as 3 or –½ to be considered as polynomials, but they can. If it makes you feel better you can write 3 as $0x^2 + 0x + 3$. (Remember that perhaps you once found it strange to call 3 a fraction until you learnt that it could be written as $\frac{3}{1}$.)

There's one polynomial for which the degree remains undefined. It's the zero constant polynomial, 0. (Some books define this degree to be $-\infty$ for certain technical reasons but I'll leave it undefined.)

The coefficient of x^n, for a polynomial of degree n, is called its **leading coefficient**. For example the leading coefficient of $3x^2 - x + 5$ is 3. But beware. The leading coefficient doesn't always come first. The leading coefficient of $2 - x^2$ is -1, not 2.

A **monic** polynomial is one where the leading coefficient is 1. Clearly every non-zero polynomial can be made monic by dividing by its leading coefficient.

Example 5: The polynomial $4x^3 - 8x + 1$ has degree 3. Its leading coefficient is 4 and so it is not monic. However it can be expressed as 4 times the monic polynomial $x^3 - 2x + ¼$.

If F is a field (for example F might be $\mathbb{Q}$, the system of rational numbers) we denote the set of all polynomials with coefficients coming from F by the symbol F[x].

Example 6: For example $\mathbb{R}[x]$ contains the polynomial $\pi x^2 - 2x + \sqrt{3}$.

§6.3. Addition and Multiplication of Polynomials

Polynomials are added (and subtracted) in the usual way – just add (or subtract) the corresponding coefficients. Multiplication is somewhat more complicated to describe abstractly, yet it just amounts to expanding the product of the expressions in the way you've always done. For example:

$$(ax^2 + bx + c)(dx + e) = adx^3 + (ae + bd)x^2 + (be + cd)x + ce.$$

With these operations the system F[x] behaves very much like a field itself, but with one very important difference. In a field nearly every number has an inverse under multiplication (in fact 0 is the only exception). Most polynomials, on the other hand, do *not* have inverses. For example, since $\frac{1}{x}$ and $\frac{1}{1-x}$ are not polynomials, the polynomials x and $1 - x$ don't have (polynomial) inverses. In fact the *only* polynomials with inverses are the non-zero constant polynomials such as –2 (whose inverse is the constant polynomial –½).

Now it *is* possible to write $\frac{1}{1-x}$ as $1 + x + x^2 + \ldots..$ but although this looks like a polynomial it has infinitely many terms while a polynomial, by definition, has only finitely many. An expression like $1 + x + x^2 + \ldots$ is called a **power series**.

The system F[x] of polynomials over a field, in fact, behaves much more like the system of integers (where only ±1 have integer inverses).

Theorem 1: For polynomials a(x), b(x) $\in$ F[x], where F is a field: deg [a(x)b(x)] = deg a(x) + deg b(x)
Proof: This follows from the fact that:

$$(a_m x^m + \ldots)(b_n x^n + \ldots) = a_m b_n x^{m+n} + \ldots,$$

and the fact that if a_m and b_n are non zero then $a_m b_n$ is also non-zero. ✋☺

Thus the degree of a product is the *sum* of the degrees. Does this remind you of something? The degree function behaves like the logarithm function.

Theorem 2: For polynomials a(x), b(x) $\in$ F[x],

$$\deg [a(x) + b(x)] \le \text{MAX}[\deg a(x), \deg b(x)].$$

There's nothing really surprising in this result except for the question as to why "less-than-or-equals" rather than just "equals". The answer is that when we add two polynomials of the same degree the leading coefficients can cancel thereby producing a polynomial of lower degree.

Example 7:
If $a(x) = 2x^2 - x + 7$ and $b(x) = -2x^2 + 4x + 1$ then

$$a(x) + b(x) = 3x + 8,$$

which has *smaller* degree than either a(x) or b(x).

§6.4. Division and Remainders

As mentioned earlier, one polynomial does not usually divide exactly into another. Like the system of integers we're usually left with a **remainder**. We get exact divisibility precisely when the remainder is zero. Furthermore this remainder, when it isn't zero, is in some sense *smaller* than whatever we are dividing by. For polynomials, *smaller* means "of smaller degree".

The process of obtaining the remainder on dividing one polynomial by another is very similar to the familiar long-division algorithm.

Example 8:

$$
\begin{array}{r l}
 & 2x + \ 4 \\ \cline{2-2}
x^2 - 2x + 7 \,) & 2x^3 + 5x - \ 3 \\
 & \underline{2x^3 - 4x^2 + 14x} \\
 & 4x^2 - \ \ 9x - \ \ 3 \\
 & \underline{4x^2 - \ \ 8x + 28} \\
 & - \ \ x - 31
\end{array}
$$

From this calculation we compute the remainder on dividing $2x^3 + 5x - 3$ by $x^2 - 2x + 7$ to be $-x - 31$. Note that the remainder has lower degree than $x^2 - 2x + 7$, the polynomial we're dividing by. Note also how we write the terms neatly underneath others of the same degree.

The result of the calculation can also be expressed as:

$$2x^3 + 5x - 3 = (x^2 - 2x + 7)(2x + 4) + (-x - 31).$$

Theorem 3 (DIVISION ALGORITHM):
If $a(x)$, $b(x)$ are polynomials and $b(x)$ is non-zero then $a(x) = b(x)q(x) + r(x)$ for some polynomials $q(x)$ and $r(x)$ where either $r(x) = 0$ or $\deg r(x) < \deg b(x)$.

The polynomial $q(x)$ is called the **quotient** and $r(x)$ is called the **remainder**.

If the remainder on dividing a(x) by b(x) is zero we say that b(x) **divides** a(x), or that a(x) is a **multiple** of b(x). If we can't be bothered saying it in words we just write $b(x) \mid a(x)$ and read it as "b(x) divides a(x)".

Example 9: Since $x^2 - 5x + 6 = (x - 2)(x - 3)$ it's true that $x - 2 \mid x^2 - 5x + 6$.
But $x^2 - 2x + 7$ doesn't divide $2x^3 + 5x - 3$.

§6.5. Substitution and the Remainder Theorem

If $f(\underline{x}) \in F[x]$, in other words f($x$) is a polynomial in x with coefficients coming from the field F, and $\alpha \in F$ we define $f(\alpha)$ to be the number, in F, that results from replacing, or **substituting**, x in the polynomial by the value α.

Example 10: If $f(x) = x^2 + x - 2$ then $f(2) = 4 + 2 - 2 = 4$, $f(0) = -2$ and $f(1) = 0$. The following theorem connects the ideas of substitution and remainder.

Theorem 4 (REMAINDER THEOREM):
The remainder on dividing f(x) by $x - \alpha$ is $f(\alpha)$.
Proof: By the Division Algorithm,

$$f(x) = (x - \alpha)q(x) + r(x)$$

for some polynomials q(x), r(x) and the remainder r(x) is either zero or has degree less than 1. In other words r(x) must be a constant polynomial, so we can drop the '(x)'

and just call it r. Now substituting $x = \alpha$ into the equation $f(x) = (x - \alpha)q(x) + r$, we get $f(\alpha) = r$.

Corollary: The polynomial $f(x)$ is divisible by $x - \alpha$ if and only if $f(\alpha) = 0$.

Example 11:

We saw that if $f(x) = x^2 + x - 2$ then $f(2) = 4$. The Remainder Theorem concludes that 4 must be the remainder on dividing $f(x)$ by $x - 2$. We also saw that $f(1) = 0$. This means that $x - 1$ is a factor of $f(x)$.

Numbers which produce zero when substituted into a polynomial $f(x)$ are just the solutions of the polynomial equation $f(x) = 0$. They are called the **zeros** of the polynomial and they're quite important features of the polynomial.

§6.6. Zeros of Polynomials

A **zero** of a polynomial $f(x)$ is a number, α, such that $f(\alpha)=0$. (Sometimes they're called the roots of the polynomial.) Solving a polynomial equation $f(x) = 0$ finding all the zeros of $f(x)$. But where do we look for potential zeros? From the coefficient field? But here we have to be a little careful.

Does the polynomial $f(x) = x^2 + 1$ have any zeros? That depends. The coefficients are real numbers so we could consider $f(x)$ as belonging to the set of real polynomials, $\mathbb{R}[x]$. If so, there are no zeros. But we can just as validly consider f(x) as belonging to the set of complex polynomials $\mathbb{C}[x]$.

The set $\mathbb{C}$ of complex numbers includes all the real numbers as well as the number, called i, with the property that $i^2 = -1$. In fact a complex number is any number of the form $a + b$i where a, b are real. Frequently we switch from one field to another. So we can say that $x^2 + 1$ has no real zeros, but two complex ones.

Now the polynomial $x - \alpha$ has degree 1 and it is called a **linear** polynomial. So, there's a connection between linear factors and zeros of a polynomial.

Theorem 5: A polynomial has a zero if and only if it has a linear factor.
Proof: Whenever we have a zero, α, we have a linear factor $x - \alpha$. Conversely having a linear factor $bx + c$ for a polynomial means that we have a zero $x = -c/b$. ✋☺

If we know one zero of a polynomial we can use the remainder theorem and divide by the corresponding linear factor. The other zeros will then be zeros of the quotient.

Example 12: Given that $x = 2$ is a zero of

$$x^3 - 7x^2 + 14x - 8,$$

find the other two zeros.
Solution: By the remainder theorem $x - 2$ is a factor. Dividing the cubic by $x - 2$ we get the other factor which we proceed to solve.

So $x^3 - 7x^2 + 14x - 8 = (x - 2)(x^2 - 5x + 4)$. The other zeros are the zeros of $x^2 - 5x + 4 = (x - 1)(x - 4)$. These other zeros are thus 1 and 4.

§6.7. Quadratic Equations

A **quadratic equation** has the form $\boldsymbol{ax^2 + bx + c = 0}$ where a, b, c are constants and where $a \neq 0$. A common method for solving quadratic equations is to factorise the left hand side.

Example 13: Solve $x^2 + 5x + 6 = 0$.
Solution: We can write $x^2 + 5x + 6$ as $(x + 2)(x + 3)$ and so we have to solve the equation
$(x + 2)(x + 3) = 0$.

Now if a product of two real numbers is zero, at least one of them must be zero.
So $x + 2 = 0$ or $x + 3 = 0$, which gives $x = -2$ and -3 as the two solutions.

The hardest part of this method is the factorising. Here, because the coefficient of x^2 is 1 we simply need to find two numbers whose sum is 5 and whose product is 6. Where the coefficient is something other than 1 it's a little more difficult.

Example 14: Solve $30x^2 - 103x + 70 = 0$.
Solution: $30x^2 - 103x + 70 = (2x - 5)(15x - 14) = 0$ so $x = 5/2$ or $14/15$.
How did we go about factorising the quadratic? We looked for factors of 30 and of 70 that combine in the right way to give 103. Perhaps $(5x + 7)(6x - 10)$? No, that gives $-8x$.
What about $(5x + 6)(6x - 14)$? No, that gives $-34x$.

It seems like we're forced to do "trial and error". It could be that the quadratic doesn't factorise nicely, with whole numbers.

Factorising a quadratic is a good method provided we can spot the factorisation quickly. But there's a general method that will always work – the **quadratic equation formula**.

Theorem 6: If $b^2 - 4ac \geq 0$, the solutions to the quadratic $ax^2 + bx + c = 0$ are:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
Proof: Suppose $ax^2 + bx + c = 0$.

Then $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. We now do something that's often called **completing the square**.

We note that a perfect square of the form:

$$(x + h)^2 = x^2 + 2hx + h^2.$$

If we let $h = \frac{b}{2a}$ then $(x + h)^2 = x^2 + \frac{b}{a}\mathrm{x} + \frac{b^2}{4a^2}$. This isn't quite the same as the left-hand side of the equation above, but it differs only in the constant term.

If we add $\frac{\mathrm{b}^2}{4\mathrm{a}^2} - \frac{c}{a}$ to both sides of that equation we get:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}.$$

Hence $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$.

Taking square roots of both sides we get:

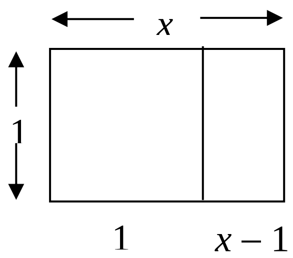
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

And so $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac < 0$ we'd need the square root of a negative number which, as far as we're concerned at this stage, doesn't exist and so there are no solutions. More correctly, there are no *real* solutions in such a case. We'll later learn about complex numbers and then we'll be able to say that there *are* solutions.

Example 15: Find the proportions of a rectangle such that if you cut off a square at one end, the left-over rectangle would have the same proportions as the original one.
Solution: Let the smaller side have length 1 and the longer side have length x.



The larger rectangle has length x and breadth 1, while the smaller rectangle has length 1 and breadth $x - 1$.

So $\frac{x}{1} = \frac{1}{x-1}$.

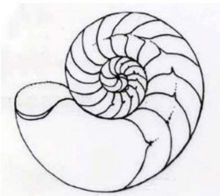
Hence $x(x - 1) = 1$ and $x^2 - x - 1 = 0$.

This doesn't factorise, but using the quadratic formula we get:

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

Now $\frac{1 - \sqrt{5}}{2} < 0$ and although it *is* a solution to the quadratic equation it clearly cannot be a solution to the original problem. So the ratio of the longer side to the shorter one for such a rectangle is $\frac{1+\sqrt{5}}{2}$.

This number, about 1.618, is called the **Golden Mean**. It's supposed to be the ideal proportion, aesthetically, for a rectangle, and many architects The builders of the Parthenon in Athens (built in the fifth century BC) used this proportion in their designs.



The golden mean also occurs in Nature, showing that the Divine Architect must have been able to solve quadratic equations!

§6.8. Sum and Product of Zeros

The **zeros** of a quadratic expression are the solutions of the corresponding equation. So, the zeros of $x^2 - 5x + 6$ are 2, 3 because the solutions of $x^2 - 5x + 6 = 0$ are 2, 3.

If α and β are the zeros of the quadratic $ax^2 + bx + c$ then we can express $\alpha + \beta$ and $\alpha\beta$ in terms of the coefficients a, b, c.

Theorem 7: If α and β are the zeros of the quadratic $ax^2 + bx + c$ then:

$$\alpha + \beta = -\frac{b}{a} \text{ and}$$

$$\alpha\beta = \frac{c}{a}.$$

Proof: While we could prove these by using the quadratic formula, the simplest proof comes from equating $ax^2 + bx + c$ to

$$a(x - \alpha)(x - \beta) = ax^2 - a(\alpha + \beta)x + a\alpha\beta.$$

Since corresponding coefficients must be equal we have $b = -a(\alpha + \beta)$ and $c = a\alpha\beta$. ✋☺

Expressions in α and β that are symmetric (this means they stay the same if α and $\beta$are swapped) can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$ and hence can be expressed easily in terms of the coefficients.

Example 16: If α, β are the zeros of the quadratic

$$x^2 - x - 2$$

find the values of:

(i) $\alpha^2 + \beta^2$;

(ii) $\frac{1}{\alpha} + \frac{1}{\beta}$;

(iii) $\alpha^2\beta + \alpha\beta^2$.

Solution: Here $\alpha + \beta = 1$ and $\alpha\beta = -2$.

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 1^2 + 4 = 5$.

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = -\frac{1}{2}$.

(iii) $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta) = -2$.

EXERCISES FOR CHAPTER 6

Exercise 1: Find the degree, the leading coefficient and the zeros of the polynomial $f(x) = 5x^6 - 6x^5 - x^7$.

Exercise 2: Which of the following are polynomials?

(a) $x^5 + x$;
(b) $x + x^{-1}$;
(c) $1 + x^2 + x^4 + \ldots..$;
(d) 42;
(e) $x^2 + \sqrt{x} + 1$.

Exercise 3: Find the quotient and remainder on dividing $f(x) = x^4 + 7x + 2$ by $x - 3$.

Exercise 4: Find the remainder on dividing

$$x^7 + x^4 + x^2 + 1$$

by $x^2 - 3$.

Exercise 5: Find the remainder on dividing

$$f(x) = x^{13} + 7x^6 - 5x^2 + x - 2$$

by $x - 1$.

Exercise 6: Find the remainder on dividing

$$x^5 - 2x^3 + 5x^2 - 7$$

by $x^2 + x + 2$.

Exercise 7: Given that 2 is a zero of

$$x^3 - 6x^2 + 9x - 2,$$

find the other two zeros.

Exercise 8: Show that 3 is a zero of the cubic

$$f(x) = x^3 - 7x^2 + 4x + 24.$$

Hence find the other two zeros.

Exercise 9: Solve the equation $\dfrac{1}{x} + \dfrac{1}{x-1} + \dfrac{1}{x+1} = 0.$

Exercise 10:

Solve the quadratic equation $x^2 - 3x - 40 = 0$.

Exercise 11: Solve the quadratic equation

$$x^2 - 2x - 40 = 0.$$

Exercise 12: If α and β are the zeros of $4x^2 - 12x + 7$ find the values of:

(i) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$;

(ii) $\alpha^2\beta + \alpha\beta^2$;

(iii) $(\alpha - \beta)^2$;

(iv) $\alpha - \beta$.

Exercise 13: If α and β are the zeros of the quadratic $2x^2 - 3x + 8$, find the values of:

(i) $\alpha^3 + \beta^3$;

(ii) $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3}$;

(iii) $\alpha\beta^4 + \beta\alpha^4$.

SOLUTIONS FOR CHAPTER 6

Exercise 1: $f(x) = -x^7 + 5x^6 - 6x^5$

$= -x^5(x - 2)(x - 3)$, so deg $f(x) = 7$; the leading coefficient is -1 and the zeros are 0, 2, 3.

Exercise 2: (a) and (d) are polynomials; the others are not.

Exercise 3:

$$
\begin{array}{rllll}
 & & x^3 + 3x^2 & + 9x & + 34 \\ \cline{2-5}
x - 3\,) & x^4 & & + 7x & + 2 \\
 & \underline{x^4 \quad - 3x^3} & & & \\
 & 3x^3 & & + 7x & + 2 \\
 & \underline{3x^3 - 9x^2} & & & \\
 & & 9x^2 & + 7x & + 2 \\
 & & \underline{9x^2 - 27x} & & \\
 & & & 34x & + 2 \\
 & & & \underline{34x - 102} & \\
 & & & & 104
\end{array}
$$

Hence the quotient is $x^3 + 3x^2 + 9x + 34$ and the remainder is 104. If we just wanted the remainder it would be much easier to use the Remainder Theorem.

The remainder is $f(3) = 81 + 21 + 2 = 104$.

Exercise 4:

$$
\begin{array}{rrrrrrrrrr}
 & & & x^5 & & +\,3x^3 & +\,x^2 & +\,9x & +\,4 \\
\cline{2-9}
x^2-3\,) & x^7\,+ & & & x^4 & & +\,x^2 & & +\,1 \\
 & x^7 & -\,3x^5 & & & & & & \\
\cline{2-9}
 & & 3x^5 & & +\,x^4 & & +\,x^2 & & +\,1 \\
 & & 3x^5 & & & -\,9x^3 & & & \\
\cline{3-9}
 & & & & x^4 & +\,9x^3 & +\,x^2 & & +\,1 \\
 & & & & x^4 & & -\,3x^2 & & \\
\cline{5-9}
 & & & & & 9x^3 & +\,4x^2 & & +\,1 \\
 & & & & & 9x^3 & & -\,27x & \\
\cline{6-9}
 & & & & & & 4x^2 & +\,27x & +\,1 \\
 & & & & & & 4x^2 & & -\,12 \\
\cline{7-9}
 & & & & & & & 27x & +\,13
\end{array}
$$

The remainder is $27x + 13$.

Exercise 5: The remainder is f(1) = 2.

Exercise 6:

$$\begin{array}{r}
x^3 \quad - x^2 - 3x + 10 \\
x^2 + x + 2 \overline{) \, x^5 \qquad - 2x^3 + 5x^2 \qquad\quad - 7} \\
\underline{x^5 + x^4 + 2x^3 \qquad\qquad\qquad\qquad} \\
- x^4 - 4x^3 + 5x^2 \qquad\quad - 7 \\
\underline{- x^4 - \; x^3 - 2x^2 \qquad\qquad} \\
- 3x^3 + 7x^2 \qquad\quad - 7 \\
\underline{- 3x^3 - 3x^2 \; - 6x} \quad \\
10x^2 + 6x - 7 \\
\underline{10x^2 + 10x + 20} \\
- 4x - 27
\end{array}$$

Exercise 7:

$$\begin{array}{r}
x^2 - 4x + 1 \\
x - 2 \overline{) \, x^3 - 6x^2 + 9x - 2} \\
\underline{x^3 - 2x^2 \qquad\qquad} \\
- 4x^2 + 9x - 2 \\
\underline{- 4x^2 + 8x \quad\;\;} \\
x - 2 \\
\underline{x - 2} \\
0
\end{array}$$

To find the other two zeros we solve the quadratic

$$x^2 - 4x + 1 = 0, \text{ getting } x = 2 \pm \sqrt{3}\,.$$

So the zeros of f(x) are 2, $2 \pm \sqrt{3}$.

Exercise 8: We could verify the fact that $x - 3$ is a factor by checking that $f\,(3) = 0$. However, since we'll need to find the quotient we must carry out the long division process.

$$
\begin{array}{r l}
 & x^2 \quad - 4x \quad - 8 \\
\cline{2-2}
x - 3 \,) & x^3 - 7x^2 + \ 4x \ + 24 \\
 & \underline{x^3 - 3x^2 \qquad\qquad\quad} \\
 & \quad - 4x^2 + \ 4x + 24 \\
 & \quad \underline{- 4x^2 + 12x \qquad} \\
 & \qquad\qquad - \ 8x + 24 \\
 & \qquad\qquad \underline{- \ 8x + 24} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

The quotient is $x^2 - 4x - 8$. Solving this quadratic we get $x = 2 \pm \sqrt{12}$, that is $2 \pm 2\sqrt{3}$.

So the zeros of f(x) are 3, $2 \pm 2\sqrt{3}$.

Exercise 9: $\dfrac{(x^2 - 1) + (x^2 + x) + (x^2 - x)}{x(x^2 - 1)} = 0$.

Hence $\dfrac{3x^2 - 1}{x(x^2 - 1)} = 0$ and so $x^2 = \dfrac{1}{3}$ or $x = \pm \dfrac{1}{\sqrt{3}}$.

Exercise 10: $(x + 5)(x - 8) = 0$, so $x = -5$ or 8.

Exercise 11: This doesn't factorise nicely so we use the quadratic equation formula.

$$x = \frac{2 \pm \sqrt{4 + 160}}{2} = -5.403 \text{ and } 7.403.$$

Exercise 12: $\alpha + \beta = 3$, $\alpha\beta = 7/4$.

(i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{22}{7}$;

(ii) $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta) = 21/4$;

(iii) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 9 - 7 = 2$;

(iv) $\alpha - \beta = \pm\sqrt{2}$.

Exercise 13: $\alpha + \beta = 3/2$, $\alpha\beta = 4$

(i) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2$

$$= \frac{27}{8} - 3\alpha\beta(\alpha + \beta)$$

$$= \frac{27}{8} - 12\left(\frac{3}{2}\right)$$

$$= -\frac{117}{8}.$$

(ii) $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\alpha^3 + \beta^3}{(\alpha\beta)^3}$

$$= \left(-\frac{117}{8}\right) \div 4^3$$

$$= -\frac{117}{512}.$$

(iii) $\alpha\beta^4 + \beta\alpha^4 = \alpha\beta(\beta^3 + \alpha^3)$

$$= 4 \times \left(-\frac{117}{8}\right)$$

$$= -\frac{117}{2}.$$